Physics 53

Rotational Motion 3

Sir, I have found you an argument, but I am not obliged to find you an understanding.

— Samuel Johnson

Angular momentum

With respect to rotational motion of a body, moment of inertia plays the same role that mass plays in the translational motion of a particle. It measures the intrinsic reluctance of a body to have its state of rotation changed.

Torque plays the role in rotation that force plays in the translational motion of a particle. It describes the external influence that causes changes in the state of rotation.

But what describes "the state of rotation" itself? For a particle, the state of translational motion is described by the linear momentum, $\mathbf{p} = m\mathbf{v}$.

The corresponding quantity for rotational motion is the **angular momentum**.

Like torque, angular momentum has meaning only with respect to some specified reference point. Also like torque, its magnitude depends on the distance from that point.

We begin with the simplest system, a single particle. Later we will generalize to systems of particles, with special interest in rigid bodies. The definition for a particle is:

Angular momentum of a particle	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$

Here $\mathbf{r}$ is the position vector of the particle relative to the reference point, and $\mathbf{p} = m\mathbf{v}$ is its linear momentum.

Some properties of $\mathbf{L}$:

- $\mathbf{L}$ is a vector, perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{p}$, and thus perpendicular to both $\mathbf{r}$ and $\mathbf{p}$.
- $\mathbf{L}$ is zero if the particle moves along the line of $\mathbf{r}$, i.e., directly toward or away from the reference point.

- The magnitude is given by $L = r_\perp p$, where $r_\perp$ is the *moment arm*, defined to be the perpendicular distance from the reference point to the line along which the particle *moves*.
- Alternatively, $L = rp_\perp$, where $p_\perp$ is the component of **p** perpendicular to **r**.
- L is a maximum, equal to rp, if **p** is perpendicular to **r**. This is the case if the particle moves (at least momentarily) in a circle about the reference point.

Like linear momentum and kinetic energy, angular momentum is an important aspect of the state of motion of a particle, especially of orbital motion around some center of force. It is also an important property of the behavior of a system of particles.

Torque as a vector

Here is the general definition of the torque of a force about a given reference point:

Torque	$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

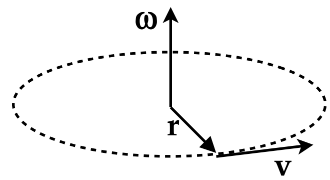
Here **r** specifies the location, relative to the reference point, of the point at which the force **F** is applied.

Some properties of $\boldsymbol{\tau}$:

- Torque is a vector, perpendicular to the plane containing **r** and **F**, and thus perpendicular to both **r** and **F**.
- The torque is zero if **F** acts along the line of **r**, i.e., directly toward or away from the reference point.
- The magnitude is given by the two formulas introduced earlier: $\tau = rF_\perp = r_\perp F$.
- The magnitude is a maximum, equal to rF, if **F** is perpendicular to **r**.

The total torque on a system of particles is the sum of the torques of all forces that act on any of the particles in the system. We will see that the torques due to internal forces cancel, so the total torque is actually the sum of the torques due only to *external* forces.

Circular motion revisited



Now we apply these new definitions to a familiar problem, that of a particle constrained to move in a circle. Shown is a particle, attached to a massless rod of length r which is pivoted at the reference point, so that the particle moves in a circle about that point. We see that $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ has magnitude $L = rmv$ and direction parallel to the angular velocity ω. Since $v = r\omega$, $L = mr^2\omega = I\omega$. In vector form $\mathbf{L} = I\boldsymbol{\omega}$.

This simple relationship between **L** and ω also holds, as we will see below, in many — but not all — important situations involving *systems* of particles. It is the rotational counterpart of $\mathbf{p} = m\mathbf{v}$.

Now consider the action of an external force applied to the particle. Let the force be tangent to the circle. (Other components would be counteracted by forces exerted by the rod.) If the force is parallel to the velocity, the particle speeds up; the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ is directed parallel to ω. Its magnitude is $\tau = rF$. This force and the torque it produces give rise to a tangential acceleration ($\mathbf{F} = m\mathbf{a}_t$) and to an angular acceleration (since $\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}$). The magnitudes obey $F = mr\alpha$, so $\tau = mr^2\alpha = I\alpha$. In vector form: $\boldsymbol{\tau} = I\boldsymbol{\alpha}$.

This simple relation between τ and α also holds in many — but not all — important situations involving *systems* of particles. It is the rotational counterpart of $\mathbf{F} = m\mathbf{a}$.

These relatively simple properties of a single particle moving in a circle carry over to the rotational motion of a rigid body — *provided it is a symmetric body rotating about its symmetry axis*. This will be shown below.

If $\mathbf{L} = I\boldsymbol{\omega}$ and $\boldsymbol{\tau} = I\boldsymbol{\alpha}$ (as in the case here) then because $\boldsymbol{\alpha} = d\boldsymbol{\omega} / dt$ (by definition) and I is a constant, we see that $\boldsymbol{\tau} = d\mathbf{L} / dt$. As we will show below, this relation is true in *all* cases. It is the rotational counterpart of $\mathbf{F} = d\mathbf{p} / dt$.

Angular momentum of a system

Here are some general properties of torques and angular momentum for any system of particles. The proofs of these statements are given at the end of this section.

Total angular momentum	$\mathbf{L}_{tot} = \mathbf{r}_{CM} \times M\mathbf{v}_{CM} + \mathbf{L}(\text{rel. to CM})$

Angular momentum — like other properties of a system — breaks up into the sum of two terms: what the angular momentum would be if the system were a single mass point at the CM, plus the angular momentum as measured in the CM frame.

Angular momentum (fixed axis)	$\mathbf{L}(\parallel \text{ to axis}) = I\boldsymbol{\omega}$

What about the components of **L** perpendicular to the axis? In general they can behave in a quite complicated way, changing with time as the body rotates. These changes must be brought about by external torques, caused by forces exerted on the body by the fixed axle about which it rotates. But if the body is *symmetric* about the axis, these perpendicular components of **L** contributed by the "mirror image" mass points on opposite sides of the axis cancel each other exactly, and no external torques are required.

For a symmetric body rotating about its symmetry axis, the angular momentum is entirely parallel to the axis and is equal to $I\omega$.

Most of the cases considered in this course involve such symmetric bodies.

An example of a body that is *not* symmetric is an unbalanced wheel on a car. The friction caused by the normal forces exerted by the axle can rapidly wear out the wheel bearings. This is why one has the wheels "dynamically balanced" to make them symmetric bodies.

Rotational 2nd law	$\boldsymbol{\tau}_{tot}^{ext} = \dfrac{d\mathbf{L}_{tot}}{dt}$

This relation holds in any inertial frame.

The reference points for the torque and for the angular momentum must be the same, of course.

One can show that this relation also holds if the reference point is chosen to be the CM, even in cases where the CM frame is non-inertial. The reason for this is that inertial forces arising in an accelerating frame act *at* the CM and thus cause no torques about that point.

From the rotational 2nd law immediately follows an important conservation law:

Conservation of angular momentum	If the total external torque on a system is zero, the total angular momentum of the system is conserved.

Some aspects of this law:

- The reference point for the torques and the angular momentum must be the same.

- Since torque and angular momentum are vectors, this law holds component by component. This means that if one component of the external torque is zero that component of the total angular momentum is conserved, even if other components are not conserved.

This is the last of the three major conservation laws of classical mechanics:

Conservation of total linear momentum. If the net external force on a system is zero, the total linear momentum of the system is conserved.

Conservation of total mechanical energy. If only conservative forces do work on a system, the total mechanical energy of the system is conserved.

Conservation of total angular momentum. If the net external torque on a system is zero about some point, the total angular momentum of the system about that point is conserved.

These laws have innumerable applications, and a working knowledge of them gives powerful insight into many physical situations.

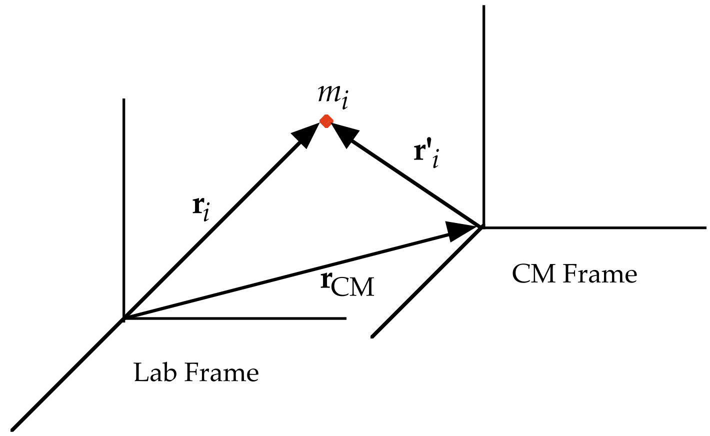
Newton's laws and these conservation laws form the core principles of this course.

As noted earlier, in classical physics there is also a law of conservation of mass, saying that the total mass of a closed system remains constant. It was noted that this law fails when relativity is taken into account, since mass can be converted into other forms of energy. Because of that, the law of conservation of mechanical energy as given above also requires modification. But the other two conservation laws retain their validity even when relativity is taken into account.

Proofs of the Statements Given Above:

Proof of $\mathbf{L}_{\text{tot}} = \mathbf{r}_{CM} \times M\mathbf{v}_{CM} + \mathbf{L}(\text{rel. to CM})$.

We use the lab and CM reference frames discussed earlier, as shown:



The particle's momentum in the lab frame is $\mathbf{p}_i = m_i\mathbf{v}_i$, where $\mathbf{v}_i = \mathbf{v}_{CM} + \mathbf{v}_i'$. Its angular momentum about the lab frame origin is therefore

$$\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i = \mathbf{r}_i \times m_i(\mathbf{v}_{CM} + \mathbf{v}_i').$$

To get the total angular momentum of the system we sum this over all the particles:

$$\begin{aligned}\mathbf{L}_{\text{tot}} &= \sum_i \mathbf{L}_i = \sum_i m_i(\mathbf{r}_{CM} + \mathbf{r}_i') \times (\mathbf{v}_{CM} + \mathbf{v}_i') \\ &= \left(\sum_i m_i\right)(\mathbf{r}_{CM} \times \mathbf{v}_{CM}) + \sum_i m_i \mathbf{r}_i' \times \mathbf{v}_i' \\ &\quad + \left(\sum_i m_i \mathbf{r}_i'\right) \times \mathbf{v}_{CM} + \mathbf{r}_{CM} \times \left(\sum_i m_i \mathbf{v}_i'\right)\end{aligned}$$

This is not as complicated as it looks:

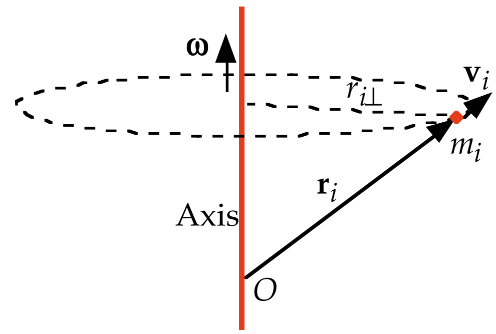
- In the middle line, the sum in the first term is just the total mass, so that term is $\mathbf{r}_{CM} \times M\mathbf{v}_{CM}$, which is the angular momentum the system would have if it were just one particle located at the CM. This is often called the "angular momentum of the center of mass."
- The second term in that line is the angular momentum as measured in the CM frame.
- In the last line, the sum in the first term is proportional to the position vector of the CM in the CM frame, while the sum in the second term is the total momentum in that frame; both of these are zero.

There are only two surviving terms, which are the two in the stated property.

There are many examples of this two-part structure. One is the motion of the earth. Its annual orbital motion around the sun gives the angular momentum *of* its CM, and its daily rotation about its axis gives the angular momentum *relative to* its CM.

Proof of $\mathbf{L}(\text{parallel to axis}) = I\boldsymbol{\omega}$.

Consider again a rigid body constrained to rotate about a fixed axis. We choose the axis to pass through the CM, which therefore does not move. The total angular momentum is thus only that *relative to* the CM.



Let the body rotate with angular velocity ω, which is a vector along the axis. We consider one of the mass points of the body, m_i. At a given instant its situation is as shown.

Its angular momentum about the origin (the CM) is

$$\mathbf{L}_i = \mathbf{r}_i \times (m_i \mathbf{v}_i).$$

This vector has components both parallel and perpendicular to the axis. The parallel component is

$$(L_i)_{\|} = m_i r_{i\perp} v_i = m_i r_{i\perp}{}^2 \omega.$$

If we sum this component over all the particles to get the total component parallel to the axis, we find

$$L_{\|} = \sum_i m_i r_{i\perp}{}^2 \omega.$$

But the sum multiplying ω is the moment of inertia about the axis. This proves the claim.

Proof of the rotational 2nd law.

The total torque acting on a system of particles is the sum of all the torques acting on the individual particles. Those torques can arise from either external or internal forces.

Earlier, when adding all the forces on the particles of a system to get the total force, we found that — because of Newton's 3rd law — the internal forces cancel each other exactly. A similar thing happens with the torques, if (as is the case for ordinary internal forces) the interaction forces between pairs of particles act along the line between the particles. As a result, the total torque on a system is just the sum of the external torques.

We showed earlier, for a single particle, that the torque about a given reference point is equal to the rate of change of the angular momentum about that reference point. We will now see that this is a general law for systems of particles.

The total angular momentum of the system is the sum of the angular momentum of the particles:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i .$$

The time derivative of this quantity is, by the product rule:

$$\frac{d\mathbf{L}}{dt} = \sum_i \mathbf{v}_i \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} .$$

The first sum on the right is zero because **v** is parallel to **p** for each particle. In the second sum, $d\mathbf{p}_i / dt = \mathbf{F}_i$, the total force on the ith particle. The second sum is thus the total torque, which (as argued above) is just the sum of the *external* torques. This proves the claim.

Static equilibrium of a rigid body

A rigid body in static equilibrium is completely at rest and remains so. The CM remains at rest, and there is no rotation about the CM. These things can happen only if both of the following are true:

- The total external force is zero.
- The total torque about the CM is zero.

The general conditions for static equilibrium are thus:

Static Equilibrium	Translational: $\mathbf{F}_{tot}^{ext} = 0$. Rotational: $\boldsymbol{\tau}_{tot}^{ext} = 0$.

These are both conditions on vectors, so they must hold for all components.

One can easily show that *any* point can be used for the reference point for the torques (not just the CM) as long as the total force is zero.

In the cases treated in this course, the forces usually all lie in a plane, so the condition on the forces involves two component equations. Any torque about a point in that plane will have only a component perpendicular to the plane, so the condition on the torques gives only one equation. Only situations with three or fewer unknowns can be completely determined by these conditions.

Stress and strain

We have usually treated forces as though they act at a single point. This is at best an approximation. Generally forces are distributed over parts of the object.

Gravity, for example, acts individually on every particle in the body. The total gravitational force is the sum of these individual forces. For a small object near the earth's surface, the force on each particle is simply $m\mathbf{g}$. If $\mathbf{g}$ is the same at the location of all the particles, the total gravitational torque about the CM of the body is easily shown to be zero. For that reason we say that gravity acts effectively at the CM of the body.

This is not true of large bodies like the earth itself, acted on by gravity from other bodies like the sun and moon, since $\mathbf{g}$ is not the same at the location of all the particles. One consequence of this is tidal forces and torques, which will be discussed later.

The normal force exerted by a surface on a body acts at all the atoms and molecules of the common interface. Friction similarly acts at all points on the interface.

To discuss the distribution of a force over a surface, one introduces the concept of force per unit area, which is called the **stress**.

Stresses are divided into three categories:

Tension acts to pull particles at the surface directly away from the body.

Compression acts to push particles at the surface directly into the body.

Shear acts to move particles at the surface parallel to that surface.

Tensile and compressive stresses involve forces *normal* to the surface of the body, while shear stresses involve forces *along* the surface.

Stresses generally produce (at least momentary) deformations of the body. These are called **strains**. If the deformation is small and not permanent, then we have a situation like Hooke's law for springs. The strain is (approximately) linearly proportional to the stress, and the constant of proportionality is called an **elastic modulus**.

In the case of tension or compression the fractional change in the dimension of the object perpendicular to the interface is related to the tensile or compressive stress by Young's modulus:

$$Y = \frac{\text{Stress}}{\text{Strain}} = \frac{F/A}{\Delta L/L}.$$

Here F/A is the force per unit area normal to the interface, and the strain is the fractional change in the dimension normal to the interface.

In the case of shear, the strain is defined to be the ratio of the lateral deformation (Δx) to the dimension perpendicular to the interface (h). The corresponding modulus is called the shear modulus:

$$S = \frac{F/A}{\Delta x/h}.$$

In this case, F/A is the ratio of the force along the interface to the area of the interface.

An object may be subject to forces acting at all points on its surface. If these forces are everywhere normal to the surface, the effect is to compress (or expand) the volume occupied by the object. In this case the force per unit area is called **pressure**:

$$P = F/A.$$

P is positive if the force on the body is inward, i.e., compressive.

If the pressure increases by ΔP, the volume V occupied by the body will decrease by ΔV. The relationship is given by the bulk modulus:

$$B = -\frac{\Delta P}{\Delta V/V}.$$

We will return to pressure when we discuss fluids.